THEORY OF SYSTEMS WITH ALTERNATION

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PMM Vol.25, No.4, 1961, pp. 691-704

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(Received April 3, 1961)

A dynamical system will be called a system with alternation if its operating conditions change periodically in such a way that the structure of the system and the applied forces have one form on the interval $0 < t < \tau_1$, and a different form on the interval $\tau_1 < t < \tau$. The process of the alternation of these conditions is repeated with a period $\tau = \text{const}$, which is called the period of alternation.

Among systems with alternation one can list pulse systems, continuous control systems, electric systems with rectifiers, electromagnetic apparatuses with saturation, and others [1-4].

To each of the intervals into which the period of alternation is divided, there corresponds some system of differential equations with variable coefficients, which change during a given period of alternation as well as from one period to the next. With the aid of these systems of equations it is possible to construct one system of differential equations with variable coefficients which is valid for any instant of time. In many problems it is, however, advantageous to make use of the above-mentioned alternating systems of differential equations, because the coefficients of these equations usually have a more simple form than those of the single equation.

1. Equations of motion. For the study of a system with alternation it is advisable to go over from alternating systems of differential equations to systems of finite difference equations which can be obtained in the following way.

Let us begin the consideration of the motion at some instant of time $n\tau + \epsilon$, where $0 \leqslant \epsilon \leqslant \tau_1$, while n is an arbitrary integer. During the time interval $n\tau + \epsilon < t < n\tau + \tau_1$, the motion is described by a system of linear differential equations with variable coefficients

$$\dot{z}_j + \sum_{k=1}^{r} b_{jk}(t)\, z_k = x_j(t) \qquad (j = 1, \ldots, r) \tag{1.1}$$

Here $z_j (j = 1, \ldots, r)$ are phase coordinates of the system, the $x_j(t)$ are given external forces. The system of scalar equations (1.1) can be replaced by the matrix equation

$$\dot{z} + b(t)\, z = x(t)$$
$$(z = \| z_j \|, \; b(t) = \| b_{jk}(t) \|, \quad x(t) = \| x_j(t) \|) \tag{1.2}$$

The solution of the matrix equation (1.2) has the following form:

$$z(t) = L(t, n\tau + \varepsilon)\, z(n\tau + \varepsilon) + \int_{n\tau+\varepsilon}^{t} L(t, \xi)\, x(\xi)\, d\xi \tag{1.3}$$
$$(L(t, \xi) = \sigma(t)\, \sigma^{-1}(\xi))$$

Here $\sigma(t)$ is the fundamental matrix for the homogeneous matrix equation obtained from (1.2) when $x(t) \equiv 0$. The inverse of this matrix we denote by σ^{-1}. The function $L(t, \xi) = \| L_{jk}(t, \xi) \|$ represents a matrix weight function for the system of differential equations (1.1).

In accordance with (1.3), the elements of the matrix $z(t)$ have the form

$$z_\mu(t) = \sum_{k=1}^{r} L_{\mu k}(t, n\tau + \varepsilon)\, z_k(n\tau + \varepsilon) + \int_{n\tau+\varepsilon}^{t} \sum_{k=1}^{r} L_{\mu k}(t, \xi)\, x_k(\xi)\, d\xi$$
$$(\mu = 1, \ldots, r) \tag{1.4}$$

At the time when $t = n\tau + \tau_1$ the phase coordinates z_μ will have the form

$$z_\mu(n\tau + \tau_1) = \sum_{k=1}^{r} L_{\mu k}(n\tau + \tau_1, n\tau + \varepsilon)\, z_k(n\tau + \varepsilon) +$$
$$+ \int_{n\tau+\varepsilon}^{n\tau+\tau_1} \sum_{k=1}^{r} L_{\mu k}(n\tau + \tau_1, \xi)\, x_k(\xi)\, d\xi \qquad (\mu = 1, \ldots, r) \tag{1.5}$$

In the next time interval $n\tau + \tau_1 < t < (n + 1)\tau$ the motion of the system will be described by the differential equations

$$\dot{z}_j + \sum_{k=1}^{r} c_{jk}(t)\, z_k = s_j(t) \qquad (j = 1, \ldots, r) \tag{1.6}$$

The system of scalar equations (1.6) is equivalent to the matrix equation

$$\dot{z} + c\,(t)\,z = s\,(t), \qquad (c\,(t) = \|c_{jk}\,(t)\|,\ s\,(t) = \|s_j\,(t)\|) \tag{1.7}$$

The differential matrix equation (1.7) has the solution

$$z\,(t) = M\,(t,\,n\tau + \tau_1)\,z\,(n\tau + \tau_1) + \int\limits_{n\tau+\tau_1}^{t} M\,(t,\,\xi)\,s\,(\xi)\,d\xi \tag{1.8}$$

Here $M(t,\ \xi) = \|\ M_{jk}(t,\ \xi)\ \|$ is a matrix weight function of the system (1.6). The elements of the matrix $z(t)$ are, by (1.8), equal to

$$z_j\,(t) = \sum_{\mu=1}^{r} M_{j\mu}\,(t,\,n\tau + \tau_1)\,z_\mu\,(n\tau + \tau_1) + \int\limits_{n\tau+\tau_1}^{t} \sum_{\mu=1}^{r} M_{j\mu}\,(t,\,\xi)\,s_\mu\,(\xi)\,d\xi$$

$$(j = 1,\,\ldots,\,r) \tag{1.9}$$

At the end of the interval considered, i.e. at the instant of time $t = n\tau + \tau$, the phase coordinates of the system will take on the values

$$z_j\,((n + 1)\,\tau) = \sum_{\mu=1}^{r} M_{j\mu}\,((n + 1)\,\tau,\,n\tau + \tau_1)\,z_\mu\,(n\tau + \tau_1) +$$

$$+ \int\limits_{n\tau+\tau_1}^{(n+1)\tau} \sum_{\mu=1}^{r} M_{j\mu}\,((n + 1)\,\tau,\,\xi)\,s_\mu\,(\xi)\,d\xi \qquad (j = 1,\,\ldots,\,r) \tag{1.10}$$

Substituting the values $z_\mu(n\tau + \tau_1)$ from (1.5) into (1.10), we reduce these relations to the following form:

$$z_j\,((n + 1)\,\tau) = \sum_{k=1}^{r} \sum_{\mu=1}^{r} M_{j\mu}\,((n+1)\,\tau,\,n\tau+\tau_1)\,L_{\mu k}\,(n\tau+\tau_1,\,n\tau+\varepsilon)\,z_k\,(n\tau+\varepsilon) +$$

$$+ \sum_{\mu=1}^{r} M_{j\mu}\,((n + 1)\,\tau,\,n\tau + \tau_1) \int\limits_{n\tau+\varepsilon}^{n\tau+\tau_1} \sum_{k=1}^{r} L_{\mu k}\,(n\tau + \tau_1,\,\xi)\,x_k\,(\xi)\,d\xi +$$

$$+ \int\limits_{n\tau+\tau_1}^{(n+1)\tau} \sum_{\mu=1}^{r} M_{j\mu}\,((n + 1)\,\tau,\,\xi)\,s_\mu\,(\xi)\,d\xi \qquad (j = 1,\,\ldots,\,r) \tag{1.11}$$

On the time interval $(n + 1)\tau < t < (n + 1)\tau + \epsilon$, which ends at the moment $(n + 1)\tau + \epsilon$ that differs from the initial time $n\tau + \epsilon$ by one period of alternation, we have the differential equations (1.1). The phase coordinates of the system will vary, in accordance with (1.4), in the following way:

$$z_\nu\,(t) = \sum_{j=1}^{r} L_{\nu j}\,(t,\,(n + 1)\,\tau)\,z_j\,((n + 1)\,\tau) + \int\limits_{(n+1)\tau}^{t} \sum_{j=1}^{r} L_{\nu j}\,(t,\,\xi)\,x_j\,(\xi)\,d\xi$$

$$(\nu = 1,\,\ldots,\,r) \tag{1.12}$$

At time $t = (n+1)\tau + \epsilon$ the phase coordinates z_ν take on the following values:

$$z_\nu((n+1)\tau+\varepsilon) = \sum_{j=1}^{r} L_{\nu j}((n+1)\tau+\varepsilon, (n+1)\tau)\, z_j((n+1)\tau) + \int_{(n+1)\tau}^{(n+1)\tau+\varepsilon} \sum_{j=1}^{r} L_{\nu j}((n+1)\tau+\varepsilon, \xi)\, x_j(\xi)\, d\xi \qquad (\nu = 1, \ldots, r) \quad (1.13)$$

By the substitution of $z_j((n+1)\tau)$ from (1.11) into (1.13), these relations can be reduced to the form

$$z_\nu((n+1)\tau+\varepsilon) + \sum_{k=1}^{r} a_{\nu k}^*(n\tau+\varepsilon)\, z_k(n\tau+\varepsilon) = X_\nu^*(n\tau+\varepsilon) \qquad (0 \leqslant \varepsilon \leqslant \tau_1 \quad \nu = 1, \ldots, r) \quad (1.14)$$

where

$$a_{\nu k}^*(n\tau+\varepsilon) = -\sum_{j=1}^{r} \sum_{\mu=1}^{r} L_{\nu j}((n+1)\tau+\varepsilon, (n+1)\tau)\, M_{j\mu}((n+1)\tau, n\tau+\tau_1) \times L_{\mu k}(n\tau+\tau_1, n\tau+\varepsilon) \quad (1.15)$$

$$\begin{aligned} X_\nu^*(n\tau+\varepsilon) = {} & \sum_{j=1}^{r} \sum_{\mu=1}^{r} L_{\nu j}((n+1)\tau+\varepsilon, (n+1)\tau)\, M_{j\mu}((n+1)\tau, n\tau+\tau_1) \times \int_{n\tau+\varepsilon}^{n\tau+\tau_1} \sum_{k=1}^{r} L_{\mu k}(n\tau+\tau_1, \xi)\, x_k(\xi)\, d\xi + \\ & + \sum_{j=1}^{r} L_{\nu j}((n+1)\tau+\varepsilon, (n+1)\tau) \int_{n\tau+\tau_1}^{(n+1)\tau} \sum_{\mu=1}^{r} M_{j\mu}((n+1)\tau, \xi) s_\mu(\xi)\, d\xi + \\ & + \int_{(n+1)\tau}^{(n+1)\tau+\varepsilon} \sum_{j=1}^{r} L_{\nu j}((n+1)\tau+\varepsilon, \xi)\, x_j(\xi)\, d\xi \end{aligned} \quad (1.16)$$

Let us now try to obtain for ϵ, on the interval $\tau_1 \leqslant \epsilon \leqslant \tau$, relations analogous to the relations (1.14).

On the time interval $n\tau + \epsilon < t < (n+1)\tau$ we have the differential equations (1.6). In accordance with (1.9) the phase coordinates z_μ will change in agreement with the law

$$z_\mu(t) = \sum_{k=1}^{r} M_{\mu k}(t, n\tau+\varepsilon)\, z_k(n\tau+\varepsilon) + \int_{n\tau+\varepsilon}^{t} \sum_{k=1}^{r} M_{\mu k}(t, \xi)\, s_\mu(\xi)\, d\xi \qquad (\mu = 1, \ldots, r) \quad (1.17)$$

At the time $t = (n + 1)\tau$ the functions $z_\mu(t)$ take on the values

$$z_\mu((n+1)\tau) = \sum_{k=1}^{r} M_{\mu k}((n+1)\tau, n\tau + \varepsilon)\, z_k(n\tau + \varepsilon) + $$
$$+ \int_{n\tau+\varepsilon}^{(n+1)\tau} \sum_{k=1}^{r} M_{\mu k}((n+1)\tau, \xi)\, s_\mu(\xi)\, d\xi \qquad (\mu = 1, \ldots, r) \qquad (1.18)$$

On the adjoining time interval $(n + 1)\tau < t < (n + 1)\tau + \tau$ we will have the differential equations (1.1), and the change of the phase coordinates will take place, in accordance with (1.4), by the following law:

$$z_j(t) = \sum_{\mu=1}^{r} L_{j\mu}(t, (n+1)\tau)\, z_\mu((n+1)\tau) + \int_{(n+1)\tau}^{t} \sum_{\mu=1}^{r} L_{j\mu}(t, \xi)\, x_\mu(\xi)\, d\xi$$
$$(j = 1, \ldots, r) \qquad (1.19)$$

At the moment $t = (n + 1)\tau + \tau_1$ the values of the phase coordinates, by (1.19) and (1.18), will be

$$z_j((n+1)\tau + \tau_1)$$
$$= \sum_{k=1}^{r} \sum_{\mu=1}^{r} L_{j\mu}((n+1)\tau + \tau_1, (n+1)\tau)\, M_{\mu k}((n+1)\tau, n\tau + \varepsilon) z_k(n\tau + \varepsilon) +$$
$$+ \sum_{\mu=1}^{r} L_{j\mu}((n+1)\tau + \tau_1, (n+1)\tau) \int_{n\tau+\varepsilon}^{(n+1)\tau} \sum_{k=1}^{r} M_{\mu k}((n+1)\tau, \xi)\, s_k(\xi) d\xi +$$
$$+ \int_{(n+1)\tau}^{(n+1)\tau+\tau_1} \sum_{\mu=1}^{r} L_{j\mu}((n+1)\tau + \tau_1, \xi)\, x_\mu(\xi)\, d\xi \qquad (j = 1, \ldots, r) \quad (1.20)$$

In the time interval $(n + 1)\tau + \tau_1 < t < (n + 1)\tau + \epsilon$ we have the differential equations (1.6), and the law of the change of the phase coordinates is, in accordance with (1.9), the following:

$$z_\nu(t) = \sum_{j=1}^{r} M_{\nu j}(t, (n+1)\tau + \tau_1)\, z_j((n+1)\tau + \tau_1) +$$
$$+ \int_{(n+1)\tau+\tau_1}^{t} \sum_{j=1}^{r} M_{\nu j}(t, \xi)\, s_j(\xi)\, d\xi \qquad (\nu = 1, \ldots, r) \qquad (1.21)$$

At the instant $t = (n + 1)\tau + \epsilon$ the values of the functions $z_\nu(t)$ are

$$z_\nu((n+1)\tau+\varepsilon) = \sum_{j=1}^{r} M_{\nu j}((n+1)\tau+\varepsilon, (n+1)\tau+\tau_1)\, z_j((n+1)\tau+\tau_1) + \\ + \int\limits_{(n+1)\tau+\tau_1}^{(n+1)\tau+\varepsilon} \sum_{j=1}^{r} M_{\nu j}((n+1)\tau+\varepsilon, \xi)\, s_j(\xi)\, d\xi \qquad (\nu = 1, \ldots, r) \tag{1.22}$$

Substituting the values $z_j((n+1)\tau + \tau_1)$ from (1.20) into (1.22), these relations become

$$z_\nu((n+1)\tau+\varepsilon) + \sum_{k=1}^{r} a_{\nu k}^{**}(n\tau+\varepsilon)\, z_k(n\tau+\varepsilon) = X_\nu^{**}(n\tau+\varepsilon) \tag{1.23}$$

$$(\tau_1 \leqslant \varepsilon \leqslant \tau,\ \nu = 1, \ldots, r)$$

Here

$$a_{\nu k}^{**}(n\tau+\varepsilon) = -\sum_{j=1}^{r}\sum_{\mu=1}^{r} M_{\nu j}((n+1)\tau+\varepsilon, (n+1)\tau+\tau_1) \times \\ \times L_{j\mu}((n+1)\tau+\tau_1, (n+1)\tau)\, M_{\mu k}((n+1)\tau, n\tau+\varepsilon) \tag{1.24}$$

$$X_\nu^{**}(n\tau+\varepsilon) = \sum_{j=1}^{r}\sum_{\mu=1}^{r} M_{\nu j}((n+1)\tau+\varepsilon, (n+1)\tau+\tau_1) \times \\ \times L_{j\mu}((n+1)\tau+\tau_1, (n+1)\tau) \int\limits_{n\tau+\varepsilon}^{(n+1)\tau} \sum_{k=1}^{r} M_{\mu k}((n+1)\tau, \xi)\, s_k(\xi) d\xi + \\ + \sum_{j=1}^{r} M_{\nu j}((n+1)\tau+\varepsilon, (n+1)\tau+\tau_1) \int\limits_{(n+1)\tau}^{(n+1)\tau+\tau_1} \sum_{\mu=1}^{r} L_{j\mu}((n+1)\tau+\tau_1, \xi) \times \\ \times x_\mu(\xi)\, d\xi + \int\limits_{(n+1)\tau+\tau_1}^{(n+1)\tau+\varepsilon} \sum_{j=1}^{r} M_{\nu j}((n+1)\tau+\varepsilon, \xi)\, s_j(\xi)\, d\xi \tag{1.25}$$

Thus, in accordance with (1.14) and (1.23), we have the following displayed relations if ϵ lies in the interval $0 \leqslant \epsilon \leqslant \tau$, which is equal to the period of alternation:

$$z_\nu((n+1)\tau+\varepsilon) + \sum_{k=1}^{r} a_{\nu k}(n\tau+\varepsilon)\, z_k(n\tau+\varepsilon) = X_\nu(n\tau+\varepsilon) \tag{1.26}$$

$$(0 \leqslant \varepsilon \leqslant \tau,\ \nu = 1, \ldots, r)$$

Here

$$a_{\nu k}(n\tau+\varepsilon)=a_{\nu k}{}^{*}(n\tau+\varepsilon)\,1\,(\tau_{1_+}-\varepsilon)+a_{\nu k}{}^{**}(n\tau+\varepsilon)\,1\,(\varepsilon-\tau_1+) \tag{1.27}$$

$$X_\nu(n\tau+\varepsilon)=X_\nu{}^{*}(n\tau+\varepsilon)\,1\,(\tau_1+-\varepsilon)+X_\nu{}^{**}(n\tau+\varepsilon)\,1\,(\varepsilon-\tau_1+) \tag{1.28}$$

$$1\,(t)=\begin{cases}0 & \text{when } t<0\\ 1 & \text{when } t\geqslant 0\end{cases}$$

The relations (1.26) connect the values of the functions $z_\nu(\nu = 1, \ldots, r)$ and the instants of time $n\tau + \epsilon$ and $(n + 1)\tau + \epsilon$, which differ from each other by one period of alternation. Here n is an arbitrary integer, while the quantity ϵ can take on any value in the interval $0 \leqslant \epsilon \leqslant \tau$.

The relations (1.26) are valid for any value of the argument

$$t=\vartheta\tau+\varepsilon \qquad \left(\vartheta=\left[\frac{t}{\tau}\right]\right) \tag{1.29}$$

where ϑ is the integer part of t/τ. These relations can, therefore, be rewritten in the form

$$z_\nu(t+\tau)+\sum_{k=1}^{r}a_{\nu k}(t)\,z_k(t)=X_\nu(t) \qquad (\nu=1,\ldots,r) \tag{1.30}$$

The relations (1.30) represent difference equations describing the given system with alternation.

In the solution of the system of difference equations (1.30) it is necessary to give a law describing the solution functions $z_j(t)$ on the time interval $0 < t < \tau$. By setting $n = \epsilon = 0$ in Expressions (1.4) and (1.9), we find that in the interval $0 < t < \tau$

$$z(t)=z^{*}(t) \tag{1.31}$$

where z^* is a matrix whose elements have the form

$$z_j{}^{*}(t)=\Big\{\sum_{k=1}^{r}\Big[L_{jk}(t,0)\,1\,(\tau_{1_+}-t)+\sum_{\mu=1}^{r}M_{j\mu}(t,\tau_1)L_{\mu k}(\tau_1,0)\times$$

$$\times\,1\,(t-\tau_{1_+})\Big]z_k(0)+\int_0^t\sum_{k=1}^{r}L_{jk}(t,\xi)\,x_k(\xi)\,d\xi\,1\,(\tau_{1_+}-t)+$$

$$+\Big[\sum_{\mu=1}^{r}M_{j\mu}(t,\tau_1)\int_0^{\tau_1}\sum_{k=1}^{r}L_{\mu k}(\tau_1,\xi)\,x_k(\xi)\,d\xi+$$

$$+ \int_{\tau_1}^{t} \sum_{\mu=1}^{r} M_{j\mu}(t, \xi)\, s_\mu(\xi)\, d\xi \Big] 1(t - \tau_{1+}) \Big\} 1(\tau - t) \qquad (j = 1, \ldots, r) \quad (1.32)$$

2. Solution of a system of linear difference equations with variable coefficients. The system of scalar difference equations (1.30) is equivalent to the matrix equation

$$z(t + \tau) + a(t)\, z(t) = X(t)$$
$$(a(t) = \| a_{\nu k}(t) \|, \ X(t) = \| X_\nu(t) \|) \qquad (2.1)$$

Let us denote by $\theta(t)$ the fundamental matrix for the homogeneous matrix equation

$$z(t + \tau) + a(t)\, z(t) = 0$$

The columns of the matrix $\theta(t)$ will be linearly independent particular solutions of this equation; the matrix $\theta(t)$ will therefore satisfy the equation

$$\theta(t + \tau) + a(t)\, \theta(t) = 0 \qquad (2.2)$$

The general solution of Equation (2.1) can be obtained by the method of variations of arbitrary constants. For this purpose we write

$$z(t) = \theta(t)\, \chi(t) \qquad (2.3)$$

where $\chi(t)$ is a column matrix to be determined. Substituting Expression (2.3) into Equation (2.1), we obtain

$$\theta(t + \tau)\, \chi(t + \tau) + a(t)\, \theta(t)\, \chi(t) = X(t)$$

or

$$\theta(t + \tau)\, [\chi(t) + \chi(t + \tau) - \chi(t)] + a(t)\, \theta(t)\, \chi(t) = X(t) \qquad (2.4)$$

Taking into account the relation (2.2), one can reduce Equation (2.4) to the form

$$\theta(t + \tau)\, \Delta\chi(t) = X(t), \qquad \text{or} \qquad \Delta\chi(t) = \theta^{-1}(t + \tau)\, X(t), \qquad (2.5)$$

where $\theta^{-1}(t)$ is the inverse of the matrix $\theta(t)$. From (2.5) it follows [5] that

$$\chi(t) = \sum_{i=1}^{\vartheta} \theta^{-1}(t + \tau - i\tau)\, X(t - i\tau) + A(t) \qquad (2.6)$$

where, in accordance with (1.29), ϑ is the integer part of t/τ, while $A(t)$ is a periodic function (of period τ) to be determined.

Making use of the change $i = \vartheta - j + 1$ of the index of summation, we transform Expression (2.6) to the form

$$\chi(t) = \sum_{j=1}^{\vartheta} \theta^{-1}(t - \vartheta\tau + j\tau) X(t - \vartheta\tau + j\tau - \tau) + A(t) \qquad (2.7)$$

Expression (2.3) will take on the form

$$z(t) = \sum_{j=1}^{\vartheta} \theta(t)\theta^{-1}(t - \vartheta\tau + j\tau) X(t - \vartheta\tau + j\tau - \tau) + \theta(t) A(t) \qquad (2.8)$$

In the interval $0 < t < \tau$ the first term on the right-hand side of (2.8) will vanish. In order that the second term on the right-hand side of (2.8) may, in accordance with (1.31), coincide with $z^*(t)$ in this time interval, it is necessary to choose for the periodic function $A(t)$ the following function:

$$A(t) = \theta^{-1}(t - \vartheta\tau) z^*(t - \vartheta\tau) \qquad (2.9)$$

where $z^*(t)$ is a matrix whose elements are determined by Expressions (1.32).

For such a choice of the periodic function $A(t)$, Expression (2.8) will take the form

$$\begin{aligned} z(t) &= \theta(t)\theta^{-1}(t - \vartheta\tau) z^*(t - \vartheta\tau) + \\ &+ \sum_{j=1}^{\vartheta} \theta(t)\theta^{-1}(t - \vartheta\tau + j\tau) X(t - \vartheta\tau + j\tau - \tau) \end{aligned} \qquad (2.10)$$

Let us introduce the function

$$N(t, j\tau) = \theta(t)\theta^{-1}(t - \vartheta\tau + j\tau) \qquad (2.11)$$

which represents a matrix weight function of the considered system of difference equations.

Expression (2.10) can be represented as

$$z(t) = N(t, 0) z^*(t - \vartheta\tau) + \sum_{j=1}^{\vartheta} N(t, j\tau) X(t - \vartheta\tau + j\tau - \tau) \qquad (2.12)$$

The elements of the matrix $z(t)$ will have the form

$$z_s(t) = \sum_{k=1}^{r} N_{sk}(t,\ 0)\ z_k{}^*\ (t - \vartheta\tau)\ +$$

$$+ \sum_{k=1}^{r} \sum_{j=1}^{\vartheta} N_{sk}(t, j\tau)\ X_k\ (t - \vartheta\tau + j\tau - \tau) \qquad (s = 1, \ldots, r) \quad (2.13)$$

Expressions (2.13) represent the solution of the matrix difference equation (2.1) which coincides with the given matrix $z^*(t)$ in the interval $0 < t < r$.

3. Determination of the weight function for a system of difference equations. The determination of the fundamental matrix $\theta(t)$, which is needed for the construction of the weight function $N(t, j\tau)$, is a quite difficult problem. For fixed values of the argument $t = t_1$, the weight function $N(t_1, j\tau)$ can be constructed with the aid of the solution of the adjoint system of difference equations, as will be shown below.

At the instant of time $t = t_1$ the solution functions $z_s(t)$ have, in accordance with (2.13), the following values:

$$z_s(t_1) = \sum_{k=1}^{r} N_{sk}(t_1, 0)\ z_k{}^*\ (t_1 - \vartheta_1\tau)\ +$$

$$+ \sum_{k=1}^{r} \sum_{i=1}^{\vartheta_1} N_{sk}(t_1, j\tau)\ X_k\ (t_1 - \vartheta_1\tau + j\tau - \tau) \qquad (s = 1, \ldots, r) \quad (3.1)$$

Expression (3.1) contains the functions $N_{sk}(t_1, j\tau)$ which are the elements of the matrix weight function for a fixed value of the argument, $t = t_1$. In order to determine them, let us consider the adjoint matrix equation

$$Z(t) + a_T(t)\ Z(t + \tau) = 0 \quad (3.2)$$

Here $a_T(t)$ is the transposed matrix of the matrix $a(t)$.

The matrix equation (3.2) is equivalent to the following system of scalar equations:

$$Z_k(t) + \sum_{l=1}^{r} a_{lk}(t)\ Z_l(t + \tau) = 0 \qquad (k = 1, \ldots, r) \quad (3.3)$$

Multiplying the μth equation ($\mu = 1, \ldots, r$) of the original system of difference equations (2.1)

$$z_{\mu}(t+\tau)+\sum_{l=1}^{r} a_{\mu l}(t)\, z_l(t)=X_{\mu}(t)$$

by $Z_{\mu}(t+\tau)$, and the νth equation ($\nu = 1, \ldots, r$) of the system (3.3) by $z_{\nu}(t)$, and adding the terms of the equations thus obtained, we derive the following relation:

$$\Delta \sum_{k=1}^{r} Z_k(t)\, z_k(t)=\sum_{k=1}^{r} Z_k(t+\tau)\, X_k(t) \tag{3.4}$$

From the relation (3.4) it follows that

$$\sum_{k=1}^{r} Z_k(t)\, z_k(t)=\sum_{i=1}^{\vartheta}\sum_{k=1}^{r} Z_k(t+\tau-i\tau)\, X_k(t-i\tau)+B(t) \tag{3.5}$$

where $B(t)$ is a periodic function to be determined.

Changing, as was done above, the summation index i by means of the formula $i = \vartheta - j + 1$, we transform the relation (3.5) to the form

$$\sum_{k=1}^{r} Z_k(t)\, z_k(t)=\sum_{k=1}^{r}\sum_{j=1}^{\vartheta} Z_k(t-\vartheta\tau+j\tau)\, X_k(t-\vartheta\tau+j\tau-\tau)+B(t) \tag{3.6}$$

In the interval $0 < t < \tau$ the first term on the right-hand side of the relation (3.6) vanishes. In order that the second term on the right of Equation (3.6) may coincide with the left-hand side of (3.6) in this interval, it is necessary to select the periodic function $B(t)$ in the following way:

$$B(t)=\sum_{k=1}^{r} Z_k^*(t-\vartheta\tau)\, z_k^*(t-\vartheta\tau) \tag{3.7}$$

where $Z_k^*(t)$ is a matrix which is defined only on the interval $0 < t < \tau$, and coincides in this interval with the matrix $Z(t)$. It is obvious that for the possibility of the construction of a solution of the adjoint matrix equation (3.2), it is necessary to know in advance the solution matrix $Z(t)$ in the interval $0 < t < \tau$.

It may turn out that it is necessary, as in the case below, to give in advance the matrix $Z(t)$ not on the initial time interval $(0, \tau)$ but on a different interval $(j\tau, (j+1)\tau)$. Because of the difference equations (3.3) such a specification determines the matrix $Z(t)$ uniquely on the initial interval $(0, \tau)$.

Substituting Expression (3.7) for $B(t)$ into (3.6), we obtain

$$\sum_{k=1}^{r} Z_k(t)\, z_k(t) = \sum_{k=1}^{r} Z_k^*(t - \vartheta\tau)\, z_k^*(t - \vartheta\tau) +$$

$$+ \sum_{k=1}^{r} \sum_{j=1}^{\vartheta} Z_k(t - \vartheta\tau + j\tau)\, X_k(t - \vartheta\tau + j\tau - \tau) \tag{3.8}$$

In accordance with what was said above, $Z_k(t) = Z_k^*(t)$ when $0 < t < r$. Hence, one may omit the asterisk on Z_k^* in the relation (3.8).

For the fixed moment $t = t_1$ the relation (3.8) will take the form

$$\sum_{k=1}^{r} Z_k(t_1)\, z_k(t_1) = \sum_{k=1}^{r} Z_k(t_1 - \vartheta_1\tau)\, z_k^*(t_1 - \vartheta_1\tau) +$$

$$+ \sum_{k=1}^{r} \sum_{j=1}^{\vartheta_1} Z_k(t_1 - \vartheta_1\tau + j\tau)\, X_k(t_1 - \vartheta_1\tau + j\tau - \tau) \tag{3.9}$$

The functions $Z_k(t)$ are solutions of the system of linear difference equations (3.3). Let us require that these solutions satisfy also the conditions

$$Z_s(t) = 1, \; Z_\mu(t) = 0 \qquad (\mu = 1, \ldots, s-1, \; s+1, \ldots, r) \tag{3.10}$$

for every value of t on the interval $\vartheta_1 r < t < (\vartheta_1 + 1)r$.

Under the conditions (3.10), the relation (3.9) takes on the form

$$z_s(t_1) = \sum_{k=1}^{r} Z_k(t_1 - \vartheta_1\tau)\, z_k^*(t_1 - \vartheta_1\tau) +$$

$$+ \sum_{k=1}^{r} \sum_{j=1}^{\vartheta_1} Z_k(t_1 - \vartheta_1\tau + j\tau)\, X_k(t_1 - \vartheta_1\tau + j\tau - \tau) \tag{3.11}$$

where the index s is fixed.

Comparing Expressions (3.11) and (3.1), one can see [6] that for a fixed s

$$N_{sk}(t_1, j\tau) = Z_k(t_1 - \vartheta_1\tau + j\tau) \qquad (k = 1, \ldots, r) \tag{3.12}$$

the matrices Z_k are solutions of the adjoint system of difference equations (3.3) which satisfy the conditions (3.10) at any time t in the interval $\vartheta_1 r < t < (\vartheta_1 + 1)r$.

4. Application of difference equations with a discrete argument. Setting $\epsilon = 0$ in the difference equations (1.26), we obtain

$$z_\nu((n+1)\tau) + \sum_{k=1}^{r} a_{\nu k}(n\tau)\, z_k(n\tau) = X_\nu(n\tau) \qquad \begin{pmatrix} n = 1, 2, \ldots \\ \nu = 1, \ldots, r \end{pmatrix} \quad (4.1)$$

In accordance with (1.15) and (1.16) we now have

$$a_{\nu k}(n\tau) = -\sum_{\mu=1}^{r} M_{\nu\mu}((n+1)\tau, n\tau + \tau_1)\, L_{\mu k}(n\tau + \tau_1, n\tau) \quad (4.2)$$

$$X_\nu(n\tau) = \sum_{\mu=1}^{r} M_{\nu\mu}((n+1)\tau, n\tau + \tau_1) \int_{n\tau}^{n\tau+\tau_1} \sum_{k=1}^{r} L_{\mu k}(n\tau + \tau_1, \xi)\, x_k(\xi)\, d\xi + \\ + \int_{n\tau+\tau_1}^{(n+1)\tau} \sum_{\mu=1}^{r} M_{\nu\mu}((n+1)\tau, \xi)\, s_\mu(\xi)\, d\xi \quad (4.3)$$

The relations (4.3) connect the values of the solution functions $z_\nu (\nu = 1, \ldots, r)$ at the time instances $n\tau$ and $(n+1)\tau$ (n is an arbitrary integer). These instances represent the initial moments of time for two successive periods of alternation. These relations, which are valid for integer values n, represent difference equations with discrete arguments. Thus, the solutions of Equations (4.1) determine sequences of values of the phase coordinates z_ν at discrete points, which are the boundaries of the periods of alternation, i.e. the instants of time $t = n\tau (n = 1, 2, \ldots)$. These solutions can be obtained by the method given in Sections 2 and 3, by replacing (4.1) by the system of difference equations

$$z_\nu(t+\tau) + \sum_{k=1}^{r} a_{\nu k}{}^\circ(t)\, z_k(t) = X_\nu{}^\circ(t) \qquad (\nu = 1, \ldots, r) \quad (4.4)$$

where $a_{\nu k}{}^\circ(t)$ and $X_\nu{}^\circ(t)$ are step functions, which for $\vartheta\tau < t < (\vartheta + 1)\tau$ retain the values $a_{\nu k}(\vartheta\tau)$ and $X_\nu(\vartheta\tau)$, respectively. Here, just as above in (1.29), we denote by ϑ the integer part of t/τ. The solution of Equations (4.4) for values of t that are multiples of the alternation period τ is given, in accordance with (2.13), by

$$z_\nu(\vartheta\tau) = \sum_{k=1}^{r} N_{\nu k}(\vartheta\tau, 0)\, z_k(0) + \sum_{k=1}^{r} \sum_{j=1}^{\vartheta} N_{\nu k}(\vartheta\tau, j\tau)\, X_k(j\tau - \tau) \\ (\nu = 1, \ldots, r) \quad (4.5)$$

In some cases, in particular, when (1.1) and (1.6) represent systems of differential equations with constant coefficients, the finding of Expressions (4.5) does not present any difficulties.

During the time interval $\vartheta\tau < t < (\vartheta + 1)\tau$ the phase coordinates z_ν will change because of (1.4) and (1.9), in accordance with the law

$$z_\nu(t) = z_\nu^{**}(t) \tag{4.6}$$

where

$$\begin{aligned} z_\nu^{**}(t) = \Big\{ \sum_{k=1}^{r} \big[L_{\nu k}(t, \vartheta\tau)\, 1(\vartheta\tau + \tau_{1+} - t) + \\ + \sum_{\mu=1}^{r} M_{\nu\mu}(t, \vartheta\tau + \tau_1)\, L_{\mu k}(\vartheta\tau + \tau_1, \vartheta\tau)\, 1(t - \vartheta\tau - \tau_{1+})\big] z_k(\vartheta\tau) + \\ + \int_{\vartheta\tau}^{t} \sum_{k=1}^{r} L_{\nu k}(t, \xi)\, x_k(\xi)\, d\xi\, 1(\vartheta\tau + \tau_{1+} - t) + \\ + \Big[\sum_{\mu=1}^{r} M_{\nu\mu}(t, \vartheta\tau + \tau_1) \int_{\vartheta\tau}^{\vartheta\tau + \tau_1} \sum_{k=1}^{r} L_{\mu k}(\vartheta\tau + \tau_1, \xi)\, x_k(\xi)\, d\xi + \\ + \int_{\vartheta\tau + \tau_1}^{t} \sum_{\mu=1}^{r} M_{\nu\mu}(t, \xi)\, s_\mu(\xi)\, d\xi \Big]\, 1(t - \vartheta\tau - \tau_1 +) \Big\} \qquad (\nu = 1, \ldots, r) \end{aligned} \tag{4.7}$$

From (4.5) and (4.6) it follows that for any time t the values of the phase coordinates will be

$$z_\nu(t) = z_\nu(\vartheta\tau) + z_\nu^{**}(t)\,[1(t - \vartheta\tau\ ) - 1(t - \vartheta\tau - \tau -)] \qquad (\nu = 1, \ldots, r) \tag{4.8}$$

5. Alternating systems of linear differential equations with constant coefficients. In this case the matrix weight functions of the systems of differential equations (1.1) and (1.4) take on the form

$$L(t, \xi) = L(t - \xi), \qquad M(t, \xi) = M(t - \xi) \tag{5.1}$$

The coefficients $a_{\nu k}(n\tau + \epsilon)$ of the difference equations (1.26), which are determined by means of Expressions (1.27), (1.15) and (1.24), are transformed in accordance with (5.1) into the form

$$\begin{aligned} a_{\nu k}(n\tau + \varepsilon) = -\sum_{j=1}^{r} \sum_{\mu=1}^{r} L_{\nu j}(\varepsilon)\, M_{j\mu}(\tau_2)\, L_{\mu k}(\tau_1 - \varepsilon)\, 1(\tau_{1+} - \varepsilon) - \\ - \sum_{j=1}^{r} \sum_{\mu=1}^{r} M_{\nu j}(\varepsilon - \tau_1)\, L_{j\mu}(\tau_1)\, M_{\mu k}(\tau - \varepsilon)\, 1(\varepsilon - \tau_1\ ) \\ (n = 1, 2, \ldots; \qquad 0 < \varepsilon < \tau) \end{aligned} \tag{5.2}$$

where

$$\tau_2 = \tau - \tau_1 \tag{5.3}$$

It follows from (5.2) that for any value of $t = \vartheta\tau + \epsilon$ where, in accordance with (1.29), ϑ is the integer part of t/τ, and ϵ lies in the interval $0 < \epsilon < \tau$, the functions $a_{\nu k}$ satisfy the condition $a_{\nu k}(t + \tau) = a_{\nu k}(t)$.

Thus, for alternating systems of linear differential equations with constant coefficients, the coefficients $a_{\nu k}(t)$ of the difference equations (1.30) are periodic functions of time with a period equal to the period of alternation τ.

The functions $X_\nu(t)$ on the right-hand sides of the difference equations (1.30) take on the form

$$X_\nu(t) = X_\nu^*(\vartheta\tau + \varepsilon)\, 1(\tau_1 + -\varepsilon) + X_\nu^{**}(\vartheta\tau + \varepsilon)\, 1(\varepsilon - \tau_1 +) \tag{5.4}$$

where, in accordance with (1.16) and (1.25)

$$\begin{aligned} X_\nu^*(\vartheta\tau + \varepsilon) = {} & \sum_{j=1}^{r} \sum_{\mu=1}^{r} L_{\nu j}(\varepsilon) M_{j\mu}(\tau_2) \int_{\varepsilon}^{\tau_1} \sum_{k=1}^{r} L_{\mu k}(\tau_1 - \zeta)\, x_k(\vartheta\tau + \zeta)\, d\zeta + \\ & + \sum_{j=1}^{r} L_{\nu j}(\varepsilon) \int_{\tau_1}^{\tau} \sum_{\mu=1}^{r} M_{j\mu}(\tau - \zeta)\, s_\mu(\vartheta\tau + \zeta) d\zeta + \\ & + \int_{\tau}^{\tau_1 + \varepsilon} \sum_{j=1}^{r} L_{\nu j}(\tau + \varepsilon - \zeta)\, x_j(\vartheta\tau + \zeta)\, d\zeta \end{aligned} \tag{5.5}$$

$$\begin{aligned} X_\nu^{**}(\vartheta\tau + \varepsilon) = {} & \sum_{j=1}^{r} \sum_{\mu=1}^{r} M_{\nu j}(\varepsilon - \tau_1) L_{j\mu}(\tau_1) \int_{\varepsilon}^{\tau} \sum_{k=1}^{r} M_{\mu k}(\tau - \zeta)\, s_\mu(\vartheta\tau + \zeta)\, d\zeta + \\ & + \sum_{j=1}^{r} M_{\nu j}(\varepsilon - \tau_1) \int_{0}^{\tau_1} \sum_{\mu=1}^{r} L_{j\mu}(\tau_1 - \zeta)\, x_\mu(\vartheta\tau + \tau + \zeta)\, d\zeta + \\ & + \int_{\tau_1}^{\varepsilon} \sum_{j=1}^{r} M_{\nu j}(\varepsilon - \zeta)\, s_j(\vartheta\tau + \tau + \zeta)\, d\zeta \end{aligned} \tag{5.6}$$

The difference equations with a discrete argument (4.1) will in this case be equations with constant coefficients. Indeed, setting $\epsilon = 0$ in Expressions (5.2), we obtain

$$a_{\nu k}(n\tau) = a_{\nu k}, \qquad a_{\nu k} = -\sum_{\mu=1}^{r} M_{\nu\mu}(\tau_2)\, L_{\mu k}(\tau_1) \tag{5.7}$$

Setting $\epsilon = 0$ in (5.4), we obtain the following expressions for the functions $X_\nu(n\tau)$, which enter into the right-hand sides of the difference equations (4.1):

$$X_\nu(n\tau) = \sum_{\mu=1}^{r} M_{\nu\mu}(\tau_2) \int_0^{\tau_1} \sum_{k=1}^{r} L_{\mu k}(\tau_1 - \zeta) x_k(n\tau + \zeta) d\zeta + \\ + \int_{\tau_1}^{\tau} \sum_{\mu=1}^{r} M_{\nu\mu}(\tau - \zeta) s_\mu(n\tau + \zeta) d\zeta \qquad (5.8)$$

We note that in the particular case when $x_k(t)$ $(k = 1, \ldots, r)$ and $s_\mu(t)$ $(\mu = 1, \ldots, r)$ are step functions which preserve their values on the intervals $(n\tau, n\tau + \tau_1)$ and $(n\tau + \tau_1, (n+1)\tau)$, respectively, the functions $X_\nu(n\tau)$, in accordance with (5.8), take on the form

$$X_\nu(n\tau) = \sum_{k=1}^{r} [e_{\nu k} x_k(n\tau) + l_{\nu k} s_k(n\tau + \tau_1)] \qquad (\nu = 1, \ldots, r) \qquad (5.9)$$

where $e_{\nu k}$ and $l_{\nu k}$ are some constant coefficients determined by the following expressions:

$$e_{\nu k} = \sum_{\mu=1}^{r} M_{\nu\mu}(\tau_2) \int_0^{\tau_1} L_{\mu k}(\tau_1 - \zeta) d\zeta, \;\; l_{\nu k} = \int_{\tau_1}^{\tau} M_{\nu k}(\tau - \zeta) d\zeta \qquad (5.10)$$

The functions $z_\nu^{**}(t)$ which determine the law of change of the phase coordinates in the time interval $\vartheta\tau < t < (\vartheta + 1)\tau$, will be, in accordance with (4.7) and (5.1), of the following form:

$$z_\nu^{**}(\vartheta\tau + \varepsilon) = \sum_{k=1}^{r} [L_{\nu k}(\varepsilon) 1(\tau_{1+} - \varepsilon) + \\ + \sum_{\mu=1}^{r} M_{\nu\mu}(\varepsilon - \tau_1) L_{\mu k}(\tau_1) 1(\varepsilon - \tau_{1+})] z_k(\vartheta\tau) + \\ + \int_0^{\varepsilon} \sum_{k=1}^{r} L_{\nu k}(\varepsilon - \zeta) x_k(\vartheta\tau + \zeta) d\zeta \, 1(\tau_{1+} - \varepsilon) + \\ + \Big[\sum_{\mu=1}^{r} M_{\nu\mu}(\varepsilon - \tau_1) \int_0^{\tau_1} \sum_{k=1}^{r} L_{\mu k}(\tau_1 - \zeta) x_k(\vartheta\tau + \zeta) d\zeta + \\ + \int_{\tau_1}^{\varepsilon} \sum_{\mu=1}^{r} M_{\nu\mu}(\varepsilon - \zeta) s_\mu(\vartheta\tau + \zeta) d\zeta \Big] 1(\varepsilon - \tau_{1+}) \qquad (\nu = 1, \ldots, r) \qquad (5.11)$$

6. On the problem of the determination of the position of a system with alternation in a phase space on the basis of the deviations of one of the phase coordinates. In many automatic control systems the optimal algorithm of control is realized on the basis of the information regarding the instantaneous position of the control system in the phase space [7,8].

It is frequently difficult to obtain such information because of the unavailability of the measurements of some phase coordinate, and at times because of the absence of knowledge regarding the position of the orientation system relative to which the position of the control system is to be determined.

In this connection there arises the necessity of the development of indirect methods for obtaining information on the position of the control system in the phase space [9]. Let us pass to the presentation of one such method for a system with alternation.

It follows from (2.13) that at the instant $t = \vartheta\tau$, where ϑ is some integer, the phase coordinate z_ν takes on the following value:

$$z_s(\vartheta\tau) = \sum_{k=1}^{r} N_{sk}(\vartheta\tau, 0)\, z_k(0) + \sum_{k=1}^{r}\sum_{j=1}^{\vartheta} N_{sk}(\vartheta\tau, j\tau)\, X_k(j\tau - \tau) \qquad (6.1)$$

Let us suppose that the phase coordinate z_s can be measured. Assuming that the initial reading is not known, we measure the deviation of the phase coordinate z_s from some arbitrarily chosen origin

$$S(\vartheta_i\tau) = S^* + z_s(\vartheta_i\tau) \qquad (i = 1, \ldots, r+1) \qquad (6.2)$$

Here S^* is the deviation of the new reading origin from the initial reading origin. Denoting by L_μ the difference between two successive measurements

$$S(\vartheta_{\mu+1}\tau) - S(\vartheta_\mu\tau) = L_\mu \qquad (\mu = 1, \ldots, r) \qquad (6.3)$$

we obtain the following relations:

$$z_s(\vartheta_{\mu+1}\tau) - z_s(\vartheta_\mu\tau) = L_\mu \qquad (\mu = 1, \ldots, r) \qquad (6.4)$$

which do not contain the unknown quantity S^*.

Substituting the values $z_s(\vartheta_{\mu+1}\tau)$ and $z_s(\vartheta_\mu\tau)$ in accordance with (6.1), we obtain a system of linear algebraic equations in the initial deviations $z_k(0)$

$$\sum_{k=1}^{r} [N_{sk}(\vartheta_{\mu+1}\tau, 0) - N_{sk}(\vartheta_\mu\tau, 0)]\, z_k(0) = \qquad (\mu = 1, \ldots, r) \qquad (6.5)$$

$$= L_\mu + \sum_{k=1}^{r}\sum_{j=1}^{\vartheta_\mu} N_{sk}(\vartheta_\mu\tau, j\tau)\, X_k(j\tau - \tau) - \sum_{k=1}^{r}\sum_{j=1}^{\vartheta_{\mu+1}} N_{sk}(\vartheta_{\mu+1}\tau, j\tau)\, X_k(j\tau - \tau)$$

With Equations (6.5) one can determine the initial values of the phase coordinates $z_k(0)$ $(k = 1, \ldots, r)$, after which one can, with the aid of Formulas (2.13), determine the values of the phase coordinates $z_\nu(t)$ $(\nu = 1, \ldots, r)$ for any time t.

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Translated by H.P.T.